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COUPLED PIEZOELECTRIC VIBRATIONS OF QUARTZ PLATES

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Abstract—The contributions of coupling with the electric field and mass of electrode coatings are taken into account in solutions of equations governing coupled thickness-shear, flexure and face-shear vibrational modes in rotated-Y-cut quartz plates.

INTRODUCTION

In a recent paper [1], a derivation was given of a set of two-dimensional equations of motion of piezoelectric crystal plates—applicable up to the frequencies of the fundamental thickness-shear modes. Solutions of the equations were exhibited for the simple thickness-shear modes (i.e. thickness-shear motions independent of the coordinates in the plane of the plate) in infinite, rotated-Y-cut quartz plates executing free vibrations without electrode coatings on the plate-faces and forced vibrations induced by an alternating voltage applied to electrode coatings on the faces. In the present paper, solutions are given for free and forced vibrations in which there is variation of displacement along one direction in the plane of the plate owing to the presence of a pair of free edges in the orthogonal direction.

Quartz is a trigonal crystal: with one axis of trigonal symmetry and, in the plane at right angles, three digonal axes 120° apart. A rotated-Y-cut is a plate with one digonal axis (designated, here, by the coordinate x_1) in the plane of the plate and with an arbitrary angle ($+35°\ 15'$ for the AT-cut, $-49°\ 12'$ for the BT-cut) between the normal to the plate and the trigonal axis. In the solutions to be considered, the dependent variables are functions of x_1 alone, the free edges of the plate are at $x_1 = \pm a$ and the motion comprises coupled thickness-shear, flexure and face-shear modes. The corresponding solution without consideration of the electric field and the mass of the electrodes is the one designated as Group A in a previous paper [2].

In the case of free vibrations of the plate without electrodes, an additional branch appears in the dispersion relation for straight-crested waves when coupling with the electric field is taken into account. However, the additional wave is a non-propagating one so that no additional resonances are found when the edge conditions are applied. Furthermore, the three remaining branches are not altered very much, either qualitatively or quantitatively, because the additional (electrical) terms, for quartz, are very small in comparisons with the mechanical terms.

In the case of forced vibrations, the uniformity of the voltage applied to the electrode coatings on the faces of the plate results in the total suppression of the additional branch in the dispersion relation for straight-crested waves. Also, the electric field affects the three surviving branches even less than in the case of free vibrations without electrodes. The effect of the mass of the electrode films on those branches is somewhat larger, but not of order greater than 10^{-2} for the usual thicknesses of electrodes; so that, again, the frequency spectrum for the bounded plate is not changed very much from that calculated in [2]. The solution does, though, yield a formula for the surface charge.

FREE VIBRATIONS

From (27), (28), (30) and (32) of [1], the equations on the dependent variables $u_2^{(0)}$, $u_3^{(0)}$, $u_1^{(1)}$ and $\phi^{(1)}$, to be solved for the case of free vibrations without electrodes, are

$$\begin{aligned}
\kappa_1 c_{56} u_{3,11}^{(0)} + \kappa_1^2 c_{66}(u_{2,11}^{(0)} + u_{1,1}^{(1)}) + \kappa_1 e_{26}\phi_{,1}^{(1)} &= \rho\ddot{u}_2^{(0)},\\
c_{55} u_{3,11}^{(0)} + \kappa_1 c_{56}(u_{2,11}^{(0)} + u_{1,1}^{(1)}) + e_{25}\phi_{,1}^{(1)} &= \rho\ddot{u}_3^{(0)}\\
\gamma_{11}u_{1,11}^{(1)} - 3b^{-2}\kappa_1[c_{56}u_{3,1}^{(0)} + \kappa_1 c_{66}(u_{2,1}^{(0)} + u_1^{(1)}) + e_{26}\phi^{(1)}] + e_{11}^{(1)}\phi_{,11}^{(1)} &= \rho\ddot{u}_1^{(1)},\\
e_{11}^{(1)}u_{1,11}^{(1)} - 3b^{-2}[e_{25}u_{3,1}^{(0)} + \kappa_1 e_{26}(u_{2,1}^{(0)} + u_1^{(1)}) - \varepsilon_{22}\phi^{(1)}] - \varepsilon_{11}\phi_{,11}^{(1)} &= 0,
\end{aligned} \tag{1}$$

with edge conditions

$$T_{12}^{(0)} = T_{13}^{(0)} = T_{11}^{(1)} = D_1^{(1)} = 0 \quad \text{on} \quad x_1 = \pm a, \tag{2}$$

where

$$\begin{aligned}
T_{12}^{(0)} &= 2b\kappa_1[c_{56}u_{3,1}^{(0)} + \kappa_1 c_{66}(u_{2,1}^{(0)} + u_1^{(1)}) + e_{26}\phi^{(1)}],\\
T_{13}^{(0)} &= 2b[c_{55}u_{3,1}^{(0)} + \kappa_1 c_{56}(u_{2,1}^{(0)} + u_1^{(1)}) + e_{25}\phi^{(1)}],\\
T_{11}^{(1)} &= \tfrac{2}{3}b^3(\gamma_{11}u_{1,1}^{(1)} + e_{11}^{(1)}\phi_{,1}^{(1)}),\\
D_1^{(1)} &= \tfrac{2}{3}b^3(e_{11}^{(1)}u_{1,1}^{(1)} - \varepsilon_{11}\phi_{,1}^{(1)}),
\end{aligned} \tag{3}$$

and $\kappa_1^2 = \pi^2/12$. Note that the face-charge $D^{(1)}$ has been set equal to zero, in the fourth of (1), as the faces of the plate are not coated with electrodes. Note, also, that the terms in $u_3^{(0)}$ and $\phi^{(1)}$ in the first of (1) and the term in $\phi^{(1)}$ in the second of (1) were inadvertently omitted in (27) and (28), respectively, of [1].

For the solution of (1), we take, omitting a factor $e^{i\omega t}$,

$$\begin{aligned}
u_2^{(0)} &= A_2 b \sin \xi x_1, & u_1^{(1)} &= A_4 \cos \xi x_1,\\
u_3^{(0)} &= A_3 b \sin \xi x_1, & \phi^{(1)} &= A_5 \cos \xi x_1.
\end{aligned} \tag{4}$$

Then, upon substituting (4) in (1), we find

$$\begin{aligned}
(\bar{\xi}^2 - 3\Omega^2)A_2 + \hat{c}_{56}\bar{\xi}^2 A_3 + \bar{\xi}A_4 + \hat{e}_{26}\bar{\xi}A_5 &= 0,\\
c_{56}\bar{\xi}^2 A_2 + (\hat{c}_{55}\bar{\xi}^2 - 3\Omega^2)A_3 + \hat{c}_{56}\bar{\xi}A_4 + \hat{e}_{25}\bar{\xi}A_5 &= 0,\\
\bar{\xi}A_2 + \hat{c}_{56}\bar{\xi}A_3 + (\hat{\gamma}_{11}\bar{\xi}^2 + 1 - \Omega^2)A_4 + (\hat{e}_{26} + \hat{e}_{11}^{(1)}\bar{\xi}^2)A_5 &= 0,\\
\hat{e}_{26}\bar{\xi}A_2 + \hat{e}_{25}\bar{\xi}A_3 + (\hat{e}_{26} + \hat{e}_{11}^{(1)}\bar{\xi}^2)A_4 - (\hat{\varepsilon}_{22} + \hat{\varepsilon}_{11}^{(1)}\bar{\xi}^2)A_5 &= 0,
\end{aligned} \tag{5}$$

where

$$\Omega^2 = \rho b^2\omega^2/3\kappa_1^2 c_{66}, \qquad \bar{\xi} = \xi b,$$

$$\begin{aligned}
\hat{c}_{55} &= c_{55}/\kappa_1^2 c_{66}, & \hat{c}_{56} &= c_{56}/\kappa_1 c_{66}, & \hat{\gamma}_{11} &= \gamma_{11}/3\kappa_1^2 c_{66},\\
\hat{e}_{25} &= e_{25}/\kappa_1^2 c_{66}, & \hat{e}_{26} &= e_{26}/\kappa_1 c_{66}, & \hat{e}_{11}^{(1)} &= e_{11}^{(1)}\, 3\kappa_1^2 c_{66},\\
\hat{\varepsilon}_{22} &= \varepsilon_{22}/\kappa_1^2 c_{66}, & & & \hat{\varepsilon}_{11} &= \varepsilon_{11}/3\kappa_1^2 c_{66}.
\end{aligned}$$

If the electric terms are omitted ($e = \varepsilon = 0$), the four equations (5) reduce to the three equations (11) in [2]—after noting that the coefficient of A_2 in the third of (11) in [2] should be $\bar{\xi}$ instead of $\bar{\xi}^2$.

The dispersion relation for straight-crested waves, obtained by setting the determinant of the coefficients in (5) equal to zero, is a quartic equation in $\bar{\xi}^2$. Thus, there are four branches instead of the three (thickness-shear, flexure and face-shear) found in (11) of [2]. The fourth (electric) branch is an imaginary one for all frequencies, representing a non-propagating mode, and hence can contribute no additional resonances to the frequency spectrum of a plate of finite length. To see what effect the electric branch has on the remaining three (mechanical) branches, it is advantageous to eliminate A_5 from (5) so as to arrive at the quartic equation in the form

$$\begin{vmatrix} B_{22}\bar{\xi}^2 - 3\Omega^2 & B_{23}\bar{\xi}^2 & B_{24}\bar{\xi} \\ B_{23}\bar{\xi}^2 & B_{33}\bar{\xi}^2 - 3\Omega^2 & B_{34}\bar{\xi} \\ B_{24}\bar{\xi} & B_{34}\bar{\xi} & \hat{\gamma}_{11}\bar{\xi}^2 + B_{44} - \Omega^2 \end{vmatrix} = 0 \tag{6}$$

where

$$B_{22} = 1 + \hat{e}_{26}^2 \varepsilon_\xi^{-1}, \qquad B_{23} = \hat{c}_{56} + \hat{e}_{25}\hat{c}_{26}\varepsilon_\xi^{-1},$$
$$B_{33} = \hat{c}_{55} + \hat{e}_{25}^2 \varepsilon_\xi^{-1}, \qquad B_{34} = \hat{c}_{56} + \hat{e}_{25}(\hat{e}_{26} + \hat{e}_{11}^{(1)}\bar{\xi}^2)\varepsilon_\xi^{-1},$$
$$B_{44} = 1 + (\hat{e}_{26} + \hat{e}_{11}^{(1)}\bar{\xi}^2)^2\varepsilon_\xi^{-1}, \qquad B_{24} = 1 + \hat{e}_{26}(\hat{e}_{26} + \hat{e}_{11}^{(1)}\bar{\xi}^2)\varepsilon_\xi^{-1},$$

and $\varepsilon_\xi = \hat{\varepsilon}_{11}\bar{\xi}^2 + \hat{\varepsilon}_{22}$.

The second terms in the B_{pq} comprise the entire contribution of the electric field; and, when they are eliminated, as may be done by setting the piezoelectric constants e equal to zero, the quartic equation reduces to the cubic $(11)_1$ of [2]. Now, for the small $\bar{\xi}$ to which the equations of motion are restricted, the second terms in the B_{pq} are all of the order of $e^2/\varepsilon c$. For example, when $\bar{\xi} = 0$,

$$B_{22} = 1 + e_{26}^2/\varepsilon_{22}c_{66}.$$

For the AT-cut of quartz,

$$e_{26} = -9{\cdot}490 \times 10^{-6}\ \mathrm{C/cm^2},$$
$$\varepsilon_{22} = 39{\cdot}816 \times 10^{-21}\ \mathrm{C^2/dyn\ cm^2},$$
$$c_{66} = 29{\cdot}013 \times 10^{10}\ \mathrm{dyn/cm^2},$$

as calculated from Bechmann's constants for α-quartz [3] by Sykes's formulas [4] for rotated-Y-cuts. Hence, for the AT-cut,

$$B_{22} = 1 + 0{\cdot}0078;$$

with similar results for the other B_{pq}. Accordingly, the presence of the electric terms has little effect on the mechanical branches of the dispersion relation for straight-crested waves and, subsequently, on the frequency spectrum of a plate of finite length.

To satisfy the four edge-conditions (2), all four solutions of the type (4), corresponding to the four roots $\bar{\xi}^2$ of (6), are required. For each $\bar{\xi}$, (5) has a set of three amplitude ratios $A_2 : A_3 : A_4 : A_5$. Let $\bar{A}_i$, $i = 1, 2, 3, 5$, be the value of A_4 for the ith root $\bar{\xi}_i$; and let

$$A_2/A_4 = \alpha_{2i}, \qquad A_3/A_4 = \alpha_{3i}, \qquad A_5/A_4 = \alpha_{5i}, \qquad i = 1, 2, 3, 5$$

for each root $\bar{\xi}_i$. Then, for the complete solution of the type (4), again omitting a factor $e^{i\omega t}$,

$$
\begin{aligned}
u_2^{(0)} &= \sum_{i=1}^{3} b\bar{A}_i \alpha_{2i} \sin \xi_i x_1 + b\bar{A}_5 \alpha_{25} \sinh \xi_5 x_1, \\
u_3^{(0)} &= \sum_{i=1}^{3} b\bar{A}_i \alpha_{31} \sin \xi_i x_1 + b\bar{A}_5 \alpha_{35} \sinh \xi_5 x_1, \\
u_1^{(1)} &= \sum_{i=1}^{3} \bar{A}_i \cos \xi_i x_1 + \bar{A}_5 \cosh \xi_5 x_1, \\
\phi^{(1)} &= \sum_{i=1}^{3} \bar{A}_i \alpha_{5i} \cos \xi_i x_1 + \bar{A}_5 \alpha_{55} \cosh \xi_5 x_1.
\end{aligned}
\tag{6}
$$

Upon substituting (6) in (3) and the result in the edge conditions (2), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{3} \bar{A}_i \bar{\alpha}_{1i} \cos \bar{\xi}_i \bar{a} + \bar{A}_5 \bar{\alpha}_{15} \cosh \bar{\xi}_5 \bar{a} = 0, \\
&\sum_{i=1}^{3} \bar{A}_i \bar{\alpha}_{2i} \cos \bar{\xi}_i \bar{a} + \bar{A}_5 \bar{\alpha}_{25} \cosh \bar{\xi}_5 \bar{a} = 0, \\
&\sum_{i=1}^{3} \bar{A}_i \bar{\alpha}_{3i} \sin \bar{\xi}_i \bar{a} - \bar{A}_5 \bar{\alpha}_{35} \sinh \bar{\xi}_5 \bar{a} = 0, \\
&\sum_{i=1}^{3} \bar{A}_i \bar{\alpha}_{5i} \sin \bar{\xi}_i \bar{a} - \bar{A}_5 \bar{\alpha}_{55} \sinh \bar{\xi}_5 \bar{a} = 0,
\end{aligned}
\tag{7}
$$

where $\bar{a} = a/b$ (i.e. the ratio of the half-length to the half-thickness of the plate) and

$$
\begin{aligned}
\bar{\alpha}_{1i} &= c_{55} \alpha_{3i} \bar{\xi}_i + \kappa_1 c_{56}(\alpha_{2i} \bar{\xi}_i + 1) + e_{25} \alpha_{5i}, \\
\bar{\alpha}_{2i} &= c_{56} \alpha_{3i} \bar{\xi}_i + \kappa_1 c_{66}(\alpha_{2i} \bar{\xi}_i + 1) + e_{26} \alpha_{5i}, \\
\bar{\alpha}_{3i} &= (1 + \gamma_{11}^{-1} e_{11}^{(1)} \alpha_{5i}) \bar{\xi}_i, \\
\bar{\alpha}_{5i} &= (e_{11}^{(1)} - \varepsilon_{11} \alpha_{5i}) \bar{\xi}_i.
\end{aligned}
\tag{8}
$$

The frequency equation is obtained by setting the determinant of the coefficients of the $\bar{A}_i$ in (7) equal to zero. For convenience of comparison with the result in [2], the equation may be expressed in the form

$$
\tilde{A}_1 \tan \bar{\xi}_1 \bar{a} + \tilde{A}_2 \tan \bar{\xi}_2 \bar{a} + \tilde{A}_3 \tan \bar{\xi}_3 \bar{a} = 0,
\tag{9}
$$

where

$$
\begin{aligned}
\tilde{A}_1 &= \beta_{31}(\beta_{12}\beta_{23} - \beta_{22}\beta_{13}), \\
\tilde{A}_2 &= \beta_{32}(\beta_{13}\beta_{21} - \beta_{23}\beta_{11}), \\
\tilde{A}_3 &= \beta_{33}(\beta_{11}\beta_{22} - \beta_{21}\beta_{12}),
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
\beta_{ij} &= \bar{\alpha}_{ij} + \bar{\alpha}_{i5} \bar{\alpha}_{5j} \bar{\alpha}_{55}^{-1} \coth \xi_5 \bar{a} \tan \bar{\xi}_j \bar{a}, \qquad i = 1, 2; j = 1, 2, 3 \\
\beta_{3j} &= \bar{\alpha}_{3j} - \bar{\alpha}_{35} \bar{\alpha}_{5j}, \qquad j = 1, 2, 3.
\end{aligned}
$$

When the electric terms are omitted, (6–10) reduce to (16–20) of [2]. Owing to the non-propagating character of the electric branch, resulting in no additional resonances, and its small effect on the mechanical branches, the frequencies obtained from (9) differ little from those obtained from (19) in [2]; so that the curves marked TS_1, F_1, and FS_1 (for thickness-shear, flexure and face-shear waves travelling parallel to the x_1 direction) in Fig. 3 of [2] would be only slightly changed and, hence, would still fit the Koga–Fukuyo data [5] shown there.

FORCED VIBRATIONS

We now consider a plate with like electrode films deposited on its two faces. An alternating voltage, with drop 2V across the thickness of the plate, is applied to the electrodes. The first order electric potential is therefore fixed throughout the plate:

$$\phi^{(1)} = b^{-1} V e^{i\omega t}. \tag{11}$$

The equation of motion (27), (28), (30) and (32) of [1] are, then,

$$\begin{aligned} \kappa_1 c_{56} u^{(0)}_{3,11} + \kappa_1^2 c_{66}(u^{(0)}_{2,11} + u^{(1)}_{1,1}) &= \rho(1+R)\ddot{u}^{(0)}_2, \\ c_{55} u^{(0)}_{3,11} + \kappa_1 c_{56}(u^{(0)}_{2,11} + u^{(1)}_{1,1}) &= \rho(1+R)\ddot{u}^{(0)}_3, \\ \gamma_{11} u^{(1)}_{1,11} - 3b^{-2}\kappa_1[c_{56} u^{(0)}_{3,1} + \kappa_1 c_{66}(u^{(0)}_{2,1} + u^{(1)}_1) + e_{26}\phi^{(1)}] &= \rho(1+3R)\ddot{u}^{(1)}_1, \\ e^{(1)}_{11} u^{(1)}_{1,11} - 3b^{-2}[e_{25} u^{(0)}_{3,1} + \kappa_1 e_{26}(u^{(0)}_{2,1} + u^{(1)}_1) - \varepsilon_{22}\phi^{(1)}] + D^{(1)} &= 0, \end{aligned} \tag{12}$$

where R is the ratio of the mass per unit area of both films to the mass per unit area of the quartz plate alone; the factors $(1 + R)$ and $(1 + 3R)$ are obtained from [6]. The surface charge on each face is $\frac{1}{3}b^2 D^{(1)}$ and κ_1 has the value given by Bleustein and Tiersten [7]:

$$\kappa_1^2 = \frac{\pi^2}{12}\left[1 + R - \frac{8}{\pi^2(1 + \varepsilon_{22} c_{66}/e_{26}^2)}\right]. \tag{13}$$

The conditions for free edges are again those in (2), but we note, from (3), that the last two conditions are identical owing to the fixed value (11) of $\phi^{(1)}$. The reduction of the number of boundary conditions from four to three is consistent with the reduction of the number of dependent variables from four ($u_2^{(0)}$, $u_3^{(0)}$, $u_1^{(1)}$ and $\phi^{(1)}$) to three: $u_2^{(0)}$, $u_3^{(0)}$, and $u_1^{(1)}$. The latter are to be obtained as solutions of the first three of (12); the fourth equation serves simply as a formula for the surface charge.

We first find a particular solution of (12) for the constant forcing term $\phi^{(1)}$:

$$u_2^{(0)} = u_3^{(0)} = 0, \qquad u_1^{(1)} = A_0, \qquad \phi^{(1)} = b^{-1}V, \qquad D^{(1)} = D_0 \tag{14}$$

(omitting a factor $e^{i\omega t}$). Substitution of (14) in (12) yields, as before [1],

$$A_0 = \frac{3\kappa_1 e_{26} V}{\rho b^3(1+3R)(\omega^2 - \varpi^2)}, \tag{15}$$

$$D_0 = \frac{3\varepsilon_{22} V(1 + e_{26}^2/\varepsilon_{22} c_{66})(1 - \omega^2/\varpi_a^2)}{b^3(\omega^2/\varpi^2 - 1)}, \tag{16}$$

where

$$\varpi^2 = \frac{3\kappa_1^2 c_{66}}{\rho b^2(1+3R)}, \tag{17}$$

$$\varpi_a^2 = \frac{3\kappa_1^2 c_{66}(1 + e_{26}^2/\varepsilon_{22} c_{66})}{\rho b^2(1+3R)}. \tag{18}$$

It may be seen, from (16), that ϖ and ϖ_a are the resonance and antiresonance frequencies, respectively, of the infinite, plated, piezoelectric plate vibrating in the thickness-shear mode.

For the complementary solution, we again take (4), but with $\phi^{(1)} = 0$. Upon substituting in the first three of (12), we find

$$\begin{aligned}(\xi^2 - 3R'\bar{\Omega}^2)A_2 + \hat{c}_{56}\xi^2 A_3 + \xi A_4 &= 0,\\ \hat{c}_{56}\xi^2 A_2 + (\hat{c}_{55}\xi^2 - 3R'\bar{\Omega}^2)A_3 + \hat{c}_{56}\xi A_4 &= 0,\\ \xi A_2 + \hat{c}_{56}\xi A_3 + (\hat{\gamma}_{11}\xi^2 + 1 - \bar{\Omega}^2)A_4 &= 0,\end{aligned} \tag{19}$$

where

$$\bar{\Omega} = \omega/\bar{\omega}, \qquad R' = (1 + R)/(1 + 3R). \tag{20}$$

Thus, the dispersion relation for straight-crested waves, obtained by setting the determinant of the coefficients of A_2, A_3, A_4, in (19), equal to zero, is

$$\begin{vmatrix} \xi^2 - 3R'\bar{\Omega}^2 & \hat{c}_{56}\xi^2 & \xi \\ \hat{c}_{56}\xi^2 & \hat{c}_{55}\xi^2 - 3R'\bar{\Omega}^2 & \hat{c}_{56}\xi \\ \xi & \hat{c}_{56}\xi & \hat{\gamma}_{11}\xi^2 + 1 - \bar{\Omega}^2 \end{vmatrix} = 0. \tag{21}$$

The three roots of (21) give the usual thickness-shear, flexure and face-shear branches found in [2], but slightly modified by the influence of the electrode mass, incorporated in R' and $\bar{\Omega}$ in (21), and by both the electrode mass and the piezoelectric effect as represented in the formulas (13) and (17) for κ_1 and $\bar{\omega}$. However, contrary to the case of free vibrations of the piezoelectric plate, there is no electric branch: it is suppressed by the uniform electric potential applied over the electrodes.

For each root ξ_i, $i = 1, 2, 3$, of (21), (19) has a pair of amplitude ratios $A_2 : A_3 : A_4$. Let $\bar{A}_i$, $i = 1, 2, 3$ be the value of A_4 for the ith root ξ_i; and let

$$A_2/A_4 = \alpha_{2i}, \qquad A_3/A_4 = \alpha_{3i}$$

for each ξ_i. Then, the complete solution may be written as

$$\begin{aligned} u_2^{(0)} &= \sum_{i=1}^{3} b\bar{A}_i \alpha_{2i} \sin \xi_i x_1,\\ u_3^{(0)} &= \sum_{i=1}^{3} b\bar{A}_i \alpha_{3i} \sin \xi_i x_1,\\ u_1^{(1)} &= A_0 + \sum_{i=1}^{3} \bar{A}_i \cos \xi_i x_1,\\ \phi^{(1)} &= b^{-1}V. \end{aligned} \tag{22}$$

The edge conditions (2) now produce only three equations:

$$\begin{aligned} \sum_{i=1}^{3} \bar{A}_i \bar{\alpha}_{1i} \cos \xi_i \bar{a} &= -\kappa_1 c_{56} A_0 - e_{25} b^{-1} V,\\ \sum_{i=1}^{3} \bar{A}_i \bar{\alpha}_{2i} \cos \xi_i \bar{a} &= -\kappa_1 c_{66} A_0 - e_{26} b^{-1} V,\\ \sum_{i=1}^{3} \bar{A}_i \gamma_{11} \xi_i \sin \xi_i \bar{a} &= 0, \end{aligned} \tag{23}$$

where

$$\begin{aligned} \bar{\alpha}_{1i} &= c_{55}\alpha_{3i}\xi_i + \kappa_1 c_{56}(\alpha_{2i}\xi_i + 1),\\ \bar{\alpha}_{2i} &= c_{56}\alpha_{3i}\xi_i + \kappa_1 c_{66}(\alpha_{2i}\xi_i + 1), \end{aligned} \tag{24}$$

and A_0 is given by (15).

For each root ξ_i of (21), for a given frequency ω, the two amplitude ratios α_{2i} and α_{3i} are obtained from (19). Then $\bar{\alpha}_{1i}$ and $\bar{\alpha}_{2i}$ are calculated from (24) and inserted in (23), which may be solved for the three $\bar{A}_i$. Thus, $u_2^{(0)}$, $u_3^{(0)}$ and $u_1^{(1)}$, in (22), are fully determined. Finally, $D^{(1)}$, and hence the surface charge, is obtained from the fourth of (12).

The frequency equation is found by setting the determinant of the coefficients of the $\bar{A}_i$, in (23), equal to zero:

$$\tilde{A}_1 \tan \xi_1 \bar{a} + \tilde{A}_2 \tan \xi_2 \bar{a} + \tilde{A}_3 \tan \xi_3 \bar{a} = 0, \tag{25}$$

where

$$\begin{aligned} \tilde{A}_1 &= \xi_1(\bar{\alpha}_{12}\bar{\alpha}_{23} - \bar{\alpha}_{22}\bar{\alpha}_{13}), \\ \tilde{A}_2 &= \xi_2(\bar{\alpha}_{13}\bar{\alpha}_{21} - \bar{\alpha}_{23}\bar{\alpha}_{11}), \\ \tilde{A}_3 &= \xi_3(\bar{\alpha}_{11}\bar{\alpha}_{22} - \bar{\alpha}_{21}\bar{\alpha}_{12}). \end{aligned} \tag{26}$$

The frequency spectrum calculated from (25) differs little from that obtained from (19) of [2]. The influence of the electric field is contained entirely in the electrical term in the expression (13) for κ_1 which, for the AT-cut of quartz, for example, is only

$$8/\pi^2(1 + \varepsilon_{22}c_{66}/e_{26}^2) = 0{\cdot}006. \tag{27}$$

The influence of the mass of the electrodes makes its appearance in the dispersion relation (21) in the terms R' and $\bar{\Omega}$ as given, in terms of the mass ratio R, by (20), (17) and (13). As R is usually of the order of 10^{-2} or less, the influence of the mass of electrodes on the frequency spectrum is small qualitatively and, quantitatively, appears mostly as an alteration of the vertical scale ratio in the usual depiction of the frequency spectrum with frequency ratio as ordinate and length to thickness-ratio as abscissa, as in Fig. 3 of [2].

For the case of free vibrations of the plate with electrode coatings on its faces, it is only necessary to set $V = 0$.

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Резюме — При решении уравнений, определяющих сопряженную волну сдвигового типа, изгиб и виды колебаний поверхностного сдвига во вращаемых вилкообразно вырезанных кварцевых листах берутся во внимание способствующие факторы взаимодействия с напряженностью электрического поля и электродными покрытиями массы.